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Summary

When individuals rank alternatives, they reveal preferences over characteristics of these alternatives. The revealed preferences will almost surely exhibit stochastic, but not absolute, consistency of choice. The random utility model provides an important framework for study of stochastic aspects of preference, and this essay adapts the random utility model to examine rankings or orderings of alternatives. Results of the analysis first outline relations of random orderings to the more extensively studied phenomenon of random selection of individual alternatives and secondly provide simple expressions in closed form for the probability that alternatives will be ranked in a given sequence. The latter results provide the basis for empirical study of random orderings through maximum likelihood techniques.

This note outlines a technical problem and its solution. The problem arises when an agency, or individual, announces for (n) distinct projects an exhaustive ranking as to project importance or priority in execution. This ranking will be a bijection which maps each project into the set of the first (n) integers, or something like:

rank:	1	11	:project number
	2	4	
	3	15	
	...	...	
	n	7	

with order of importance decreasing as rank increases. Presumably, this ranking is based on the attributes of the various projects, e.g. the extent to which each project furthers stated agency goals, project cost, nature of political opposition generated, past agency experience, and so on. A basic question is how the announced ranking relates to these project attributes and to what extent (if at all) each individual attribute contributes to a high or low ranking for specific projects.

Before the technical details of this question are addressed, it seems best to highlight certain aspects of the problem:

(1) In general, not all project attributes will be observable or even quantifiable, and it will not be possible to completely explain the ranking in terms of only those attributes which can be observed.

The unobserved attributes should be scattered over the listed projects in a fashion which is essentially random. From the vantage of the outside observer then, the announced rank for each project is a random function of observed attributes. The goal of analysis is thus to estimate the importance of various attributes and to predict future rankings, using observable attributes.

(2) The integers representing project ranks are of course ordinal and not cardinal data. In other words, the ranks only imply that one project is more important than another, and do not at all denote magnitudes of differences in importance. Any monotone increasing transformation of the ranking will yield an ordering with different (possibly non-integer) values for the range, but which preserves the sequence of projects and hence is as valid a representation of priority as the original integer ranking. The restriction of the range of rankings to the first (n) positive integers represents merely a useful and common normalization. It is clearly inappropriate to regard this simple convention as conveying specific information as to absolute size of project importance. The upshot of these arguments is that any estimation technique which relies on the cardinality of dependent variables is inadequate for analysis of the problem as posed.

(3) The announcement of the ranking will in general not be acompanied by specific details of the determination of project ranks. Obviously, these details would be useful for analysis, but in their absence, modeling of the choice of ranking will be required. Models usd to explain such choice should definitely be characterized by two

aspects: a) they should be based on a satisfactory theory of decision which leads to consistent and rational choie, and b) they should explicitly recognize that the basis for empirical analysis is the observed ranking or ordering taken as a whole, and not (as is more common) recorded selection of individual projects.

I. The Random Utility Model

The arguments above characterize a ranking as a sequence or listing of projects in terms of importance to the agency, where "importance" is a random function of observable attributes. More explicitly, the importance of a project may be expressed in terms of a "usefulness index" or:

$$(1) \qquad U_j = V(z_j) + e(z_j) \qquad \text{for all } (j) \text{ in } A$$

where: z_j = vector of observable attributes for project (j)

e_j = scalar random variable (with mean independent of z_j) for project (j)

A = set of (n) distinct projects

If agency choices are made on the basis of these usefulness indices, then the agencies behavior may be said to obey the random utility model. To simplify the model for empirical work, two restrictions will be imposed on (1). First, the general function $V(z_j)$ will be approximated by a linear combination of scalar functions of observable attributes (such as logarithms, inverses, and polynomials of attributes in manner common to econometrics). Secondly, and more significantly, the random terms in (1) will be regarded as not dependent on values of the observable attributes (as would occur if this error term varied with the "similarity" among projects) and further that the random terms are identically distributed and independent of each other. This more restrictive version of the random utility model (denoted the independently, identically distributed random utility or IIDRU model) may be written as:

$$(2) \qquad U(z_j, e_j) = u(z_j) = b^t x_j + e_j \qquad \text{for all (j) in A}$$

where: x_j = vector of scalar functions of observable attributes for project (j)[1]

b = vector of policy weights for components of x_j

The error term will not be explicitly included as an argument in the utility index for subsequent convenience.

Under the random utility model (in both general and restrictive forms) an agency will choose one project for execution out of set A, denoted S(j;A), if:

$$(3) \qquad U(z_j) > U(z_k) \qquad \text{for all } k=j \text{ in } B \subset A$$

Further, the agency will choose a particular exhaustive ranking for set A if:

$$(4) \qquad U(y_1) > U(y_2) > \ldots > U(y_n)$$

where (r) denotes the ranking and (r_k) denotes the k^{th} project within the ranking or $(r_1\ r_2\ r_3\ \ldots\ r_n) = (r)$, and where $(y_{r_k} = z_k)$.[2] As an example, if project number 15 was ranked third out of all projects, its attributes would be denoted either as vector z_{15} or as vector y_3. Further, as x_k denotes a vector of functions of attributes for z_k, so also w_k will denote an equivalent vector of functions of attributes for y_k. In other words, $(w_{r_k} = x_k)$.[3]

As argued earlier, the e_j terms may be treated as random variables so that the U_j terms are random as well. Thus the above inequalities will not hold with unitary probability. Using the IIDRU model, the probability that inequality (3) holds becomes:

(5) $$\Pr S(j;B) = \Pr(U(z_j) > U(z_k) \text{ for all } k \text{ in } B)$$

$$= \Pr(b^t x_j + e_j > b^t x_k + e_k \text{ for all } k \text{ in } B)$$

$$= \int_{-\infty}^{\infty} f(v - b^t x_j) \prod_{\substack{k\varepsilon B\\ k\neq j}} F(v - b^t x_k)dv$$

where $\Pr(e < v)=F(v)$ and $f(v)$ is the associated density function.[4] Also, under the IIDRU model, the probability of the ordering in (4) becomes:[5]

(6) $$\Pr(r) = \Pr(U(y_1) > U(y_2) > U(y_3) \ldots\ )$$

$$= \Pr(b_1^t w_1 + e_1 > b_2^t w_2 + e_2 > \ldots > b_n^t w_n + e_n)$$

$$= \int_{\infty}^{\infty} f(v_1 - b_1^t w_1) \prod_{k=2}^{n} F(v_{k-1} - b_k^t w_k) \prod_{i=0}^{n-1} dv_{n-i}$$

Several observations should be made at this point:

a) The probabilities in (5) are generally called choice probabilities as they involve choice of individual projects from project sets. However, arguments in (6) are also choice probabilities as they involve choice of one particular ranking from the (n!) possible rankings of (n) projects. To distinquish these different sorts of choices, arguments in (5) will be called selection probabilities and arguments in (6) will be denoted ranking probabilities.

b) The assumption of independence for the error temrs in equation (1) is really quite strong. In particular, when projects are similar in terms of attributes, there are well known implausibilities associated with choices under equation (5) (Domemchich and McFadden, Manski). Thus analysis of either selection or ranking probabilities using the IIDRU model should be performed for projects that are conceptually at least quite distinct.

c) Empiricial applications of the IIDRU model require specification of distributions for the error terms in (2), (5), and (6). The remaining sections of this note will examine the nature and interelationships of selection and ranking probabilities derived in this fashion. While many of the results considered here have been presented elsewhere, a distinctive feature of this effort is the presentation of selection and ranking probabilities within the unifying framework of IIDRU models. In the next section, the most useful error distribution for these models, the extreme value or Weibull, will be considered.

III. The Extreme Value Distribution

A particularly useful error distribution for specification of selection and ranking probabilities is that of the extreme value or Weibull:

(7) $$\Pr(e < v) = \exp(-\exp(-av - c))$$

where $(a>0)$ and (c) are scalar constants. This distribution derives its name as it is a limiting or asymptotic distribution of the maximum of a sample of (n) independently, identically distributed random variables, as (n) goes to infinity. In other words, just as the Central Limit Theorem assurs us that the asymptotic distribution of the <u>average</u> of a sample of (n) independently, identically distributed random variables approaches the normal, so another theorem provides that the <u>maximum</u> of this sample may be distributed as in (7). The only qualifications to this latter result are that a) as with the Central Limit Theorem, not every arbitrary parent distribution has a limiting distribution, and b) there are actually three possible limiting distributions for the maximum of a sample (the exponential, the reverse exponential, and the extreme value) though only the extreme value distribution has positive density throughout its domain

(David).

Routine computation indicates that if the error terms in (2) are all distributed as in (7), then the formula for selection probabilities in (5) reduces to:

$$(8) \qquad \Pr S(j;B) = \Pr(U(z_j) > \max(U(z_k) \text{ for } k \text{ in } B)$$

$$= \frac{\exp(d^t x_j)}{\sum_{k \in B} \exp(d^t x_k)}$$

where the vector (d) is the product of the scalar (a) and the vector of policy weights (b). In similar fashion, evaluation of the formula for ranking probabilities in (6) using the extreme value distribution for the error terms in (2) and (6) yields:

$$(9) \qquad \Pr(r) = \Pr(U(y_1) > U(y_2) > \ldots > U(y_n))$$

$$= \frac{\prod_{k=1}^{n} \exp(d^t w_k)}{\prod_{k=1}^{n} \left(\sum_{j=k}^{n} \exp(d^t w_j)\right)}$$

Equations (8) and (9) actually have simple interpretations. As regards (8), consider an urn where balls of (n) different colors are placed therein. If there are w(j) balls for the j_{th} color, then the probability of drawing any particular color (k) out of urn B is:

$$\text{(10)} \qquad \Pr S(k;B) = \frac{w(k)}{\sum_{j\varepsilon B} w(j)}$$

When selection is a (possibly nonpositive) function of more than one attribute, equation (10) generalizes into:

$$\text{(11)} \qquad \Pr S(k;B) = \frac{\exp(v(z_k))}{\sum_{i\varepsilon B} \exp(v(z_i))}$$

where the $V(z_k)$ terms are as in equation (1). Selection probabilities defined by equation (11) are said to obey the strict utility model.

As for the ranking probabilities in (9), they may be straightforwardly expressed in terms of the selelction probabilities in (8) as:

$$\text{(12)} \qquad \Pr(r) = \Pr S(r_1;r_1,r_2,\ldots r_n) * \Pr S(r_2;r_2,r_3,\ldots r_n)$$
$$* \Pr S(r_3;r_3,r_4,\ldots r_n) * \ldots * \Pr S(r_{n-1};r_{n-1},r_n)$$

In other words, the probability of choosing the given ranking is equivalent to the probability that projects were recursively selected without replacement from the project set as a whole. It must be recognized that both the "urn" analogy and the "recursive selection" analogy represent properties effectively unique to choice probabilities based on the extreme value distribution. If other distributions are used for evaluation of (5) or (6), then neither (11) nor (12) respectively will hold. In fact, the following logical relationships have been demonstrated elsewhere:

$(1)^*$ and (7) if and only if (11) (McFadden, Yellott)

(11) if and only if (12) (Block and Marschak)

* with independent error terms differing only in mean.

By obvious inference, when the independent random uitlity model holds, the useful and intuitive decomposition in (12) of the ranking probabilities into the product of recursive choice probabilities is valid <u>if and only if</u> the IID error terms for the random uitlities in (2) have the extreme value distribution.

In addition to being mathematically tractable, selection probabilities based on the IIDRU model with the extreme value distribution have several desirable properties. The two most important of these properties are stochastic independence of irrelevant alternatives (SIIA) and invariance under uniform expansion of alternatives (IUEA). The former property is defined as invariance of the selection odds for two projects when changes are made in the total project set (with both projects in question remaining in the set). The selection probabilities in equation (11) clearly possess this property as:

$$(13) \qquad \frac{\Pr\ S(j;A)}{\Pr\ S(k;A)} = \frac{\Pr\ S(j;B)}{\Pr\ S(k;B)} = \frac{\exp(d^t x_j)}{\exp(d^t x_k)}$$

An important result here is that the SIIA property is not only necessary for selection probabilities to be as in equation (11), but is also sufficient or:

$$(13) \text{ if and only if } (11) \qquad \text{(Marschak, McFadden)}$$

Given the independence of error terms in (2), SIIA would seem desirable as a description of rational choice. For if projects are indeed completely distinct, there is little reason for the odds of selection to be affected by variations of project sets. Under more general circumstances, however, such as pronounced similarity of several projects, this property is quite suspect.

An even more interesting characteristic of selection probabilities in (11) has recently been demonstrated by Yellott. Suppose the given project set could be doubled in size by adding one each of exactly the projects currently in the project set. The new project set (denoted A^2) would consist of $(2n)$ projects for which every project would have one (and only one) identical twin. This sort of uniform expansion can of course be carried on (k) separate times yielding larger project sets denoted (A^k). The property of invariance under uniform expansion of alternatives (IUEA) requires that:

$$(14) \qquad \Pr S(j;A) = \Pr S(j \quad J(k);A^k) \text{ for all } k$$

where $J(k)$ is the set of (k) projects identical to (j) in the k^{th} uniform expansion of A. As an example of this property, consider a passenger on an airplane confronted with a cup of coffee, a cup of tea, and a cup of milk. Now consider the same passenger confronted with (k) identical cups of coffee, (k) identical cups of tea, and (k) identical cups of milk. What the IUEA property in (14) requires is the intuitively reasonable result that the probability of the passenger choosing a cup of coffee from the first three-object choice set is equilivant to the probability of this same passenger choosing a cup of coffee (though not a <u>specific</u> cup of coffee) from the expanded choice set. Surely the IUEA property cauptures what most of us would regard as an essential aspect of rational choice. What is so striking about this property is that, in addition to being so appealing, it is so restrictive for:

(1)* and (7) if and only if (14) (Yellott)

* with independent error terms differing only in mean.

In other words, of all IIDRU models, only those with error terms having the extreme value distribution will conform to IUEA. In light of this result, Duncan Luce has remarked that "in many contexts Yellott's condition is so compelling that his theorem means" that IIDRU models and selection probabilities in (11) "stand or fall together." (Luce).

In sum, use of the extreme value distribution for the error terms of the IIDRU model yeilds selection and ranking probabilities which possess both mathematical tractabitlity and which capture our intuitions as to the nature of rational choice. As will be seen in the following sections, these features make IIDRU models based on the extreme value distribution almost unique.

IV. A Second Error Distribution

In general, ranking probabilities are representable only by complex mathematical expressions. Using the extreme value distribution, however, ranking probabilities decompose into products of selection probabilities; in other words, there is a simple analogy for the ranking terms. Of course, other simple analogies for ranking probabilities exist, and correspond to different error distributions. For example, it is plausible to regard a ranking as having been generated through a process where the worst project was set aside and given rank (n), then the next worst project was thrown out and given rank (n-1), and so forth. This process of sequential rejection without replacement may be expressed as follows. Denote rejection probabilities as:[6]

(15) $$\Pr(U_j < \min(U_k) \text{ for all } k \text{ in } A) = \Pr R(j;A)$$

Then the probability of a ranking generated by the above-described process is:

(16) $$\Pr(r) = \Pr R(r_2;r_1,r_2) * \Pr R(r_3;r_1,r_2,r_3)$$
$$* \ldots * \Pr R(r_{n-1};r_1,r_2,\ldots r_{n-1})$$
$$* \Pr R(r_n;r_1,r_2,\ldots r_n)$$

This decomposition of ranking probabilities will hold if and only if the error term in (1) is destributed as:

(17) $$\Pr(e < v) = 1 - \exp(-\exp(av + c))$$

where (a > 0) and (c) and scalar constants. This distribution may be termed the minimal extreme value distribution, as analogously to the extreme value distribution in (7), equation (17) gives the asymptotic distribution for the <u>minimum</u> of a sample from IID random variables (subject to qualifications similar to those for the maximal extreme value distribution) (David). Using (17), the expression for rejection probabilities in (15) may be simplified to:

$$\text{(18)} \qquad \Pr R(j;B) = \frac{\exp(-d^t x_j)}{\sum_{k \in B} \exp(-d^t x_k)}$$

More importantly, using the minimal extreme value distribution in (17), the equation for ranking probabilities in (6) can be written as:

$$\text{(19)} \qquad \Pr(r) = \frac{\prod_{j=1}^{n} \exp(-d^t w_j)}{\prod_{j=1}^{n} \left(\sum_{i=1}^{n-j} \exp(-d^t w_i) \right)}$$

By inspection, equation (16) does in fact hold for ranking probabilities based on the IIDRU model and (17).

This result sheds light on a problem earlier raised by Luce and Suppes. These authors considered the comparability of ranking decompositions in (12) and (16). Specifically, they argued that if a common random utility model underlay selection, rejection, and ranking probabilities such that the ranking probabilities in (12) and (16) were equal, then necessarily:

$$b^t x_j = b^t x_k \qquad \text{for all } j \text{ and } k \text{ in } A \tag{20}$$

The theorem establishing this result is labled an "impossibility theorem" as the result in (20) cannot hold generally; thus (12) and (16) cannot in general be equal. Luce and Suppes worry that their theorem provides "powerful evidence against" expressions (12), hence (11), as reasonable descriptions of rational choice. These worries are based on the implicit assumption (mentioned above and acknowledged by Luce and Suppes) that a common random utility model actually underlies both selection and rejection probabilities, and also equations (12) and (16). Of course, as has been established here, the probabilitistic choice models leading to (12) and (16) are quite different. Thus the theorem in question only establishes the fact that ranking probabilities based on different random utility models will in general be different, except in the rare case that (20) holds. While this result is interesting, it is not a basis for criticism of the choice models examined here.

The concerns of Luce and Suppes do however expose an area of more substantive criticism. What their theorem (inadvertently) involves is a comparison of observable properties of IIDRU models based on (7) and (17) respectively. But of course, it is possible to make additional comparisons of this sort. For example, the property of SIIA may be reformulated for rejection probabilities as follows:

$$\frac{\Pr R(j;A)}{\Pr R(k,A)} = \frac{\Pr R(j;B)}{\Pr R(k;B)} \tag{21}$$

In other words, that the odds of rejection for two projects are invariant to changes in the total project set. Likewise, Yellott's IUEA property may be expressed in terms of rejections probabilities as:

$$(22) \qquad \Pr R(j;A) = \Pr R(j \quad J(k);A^{k}) \text{ for all } k \quad .$$

where the terms in (22) are as defined in (14). Equation (22) simply requires that the probability of <u>rejecting</u> one project identical to (j) does not depend on the number of such identical projects present in the uniform expansions of project set A. For convenience, denote SIIA for selection probabilities (or (13)) as SIIA-S and denote IUEA for selection probabilities (or (14)) as IUEA-S. These same properties for rejection probabilities can be expressed as SIIA-R and IUEA-R respectively for equations (21) and (22).

An important result here is that SIIA-R and IUEA-R hold if and only if the IIDRU model has the error distribution in (17). Further, both SIIA-R and IUEA-R are necessary and sufficient for rejection probabilites to be written as in (16), given the IIDRU model. The obvious upshot of these results is that SIIA-S and SIIA-R may simultaneously hold if and only if the rare condition in (20) is valid; likewise IUEA-S and IUEA-R are compatible only if (20) holds.

These "impossiblity" results appear a bit more serious that those discussed by Luce and Suppes, and suggest a need for tempering of Luce's enthusiasum for Yellot's theorem. Ceraintly both SIIA and IUEA are nifty properties which appear to represent intuitive conceptions of rational choice. But surely there is little reason to regard these restrictions on selection probabilities as any more of a hallmark of rationality that exactly the same restrictions on rejection probabilities. Since both sets of restrictions cannot simultaneously hold for IIRDU models, it is not then clear how definitive SIIA and IUEA can be as a basis for prefering one error distribution over another. Most explicitly, these properties do not seem to distinguish Error distributions in (7) and (17) on the basis of consistency with rationality. Maybe one set of restrictions is better than none, but it is nonetheless disturbing that so few of out intuitive notions concerning rational choice are fully compaible with each other. Under these circumstances, derivation of "prefered" IIDRU models using such restrictions is of uncertain usefulness.

V. The Normal Error Distribution

A common distribution for error terms in equation (2) is the normal distribution, or:

$$(24) \qquad \Pr(e < v) = \int_{-\infty}^{v} \frac{\exp(-x^2/2)}{\sqrt{2\pi}} dx$$

As argued elsewhere (Domemcich and McFadden), the selection probabilities in (4) for the IIDRU model with normally distributes error terms can be written as:

$$(25) \qquad \Pr S(j;A) = F(0;m,P)$$

where: F = cumulative multivariate (n-1) term normal function evaluated at the zero vector

m = mean vector (n-1 x 1) for the normal distribution with

Arguments $(b^t_j x - b^t_k x)$ for all k in A,

k not equal to j

P = covariance matrix (n-1 x n-1) for the

normal distribution with arguments

$p_{ij} = (2 \text{ if } i=j \text{ and } 1 \text{ if } i \neq j)$

The ranking probabilities in (5), using the normally distributed error term become:

(26) $\quad \Pr(r) = F(0;s,Q)$

where: F = as above (again n-1 term)

s = mean vector (n-1 x 1) for the

normal distribution with arguments

$$s_j = \sum_{i=1}^{j} (-1)^{i+j}\, b^t w_i$$

where j = 2, 3, ... n

Q = covariance matrix (n-1 x n-1) for the

normal distribution with arguments

$$s_{ij} = (-1)^{i+j} \min(i,j) \quad \text{where } i,j = 2, 3, \ldots n$$

The principle drawback to use of these choice probabilities is their mathematical complexity. Further, there is little intuitive rationale for either (25) or (26), and specifically (26) does not decompose in any natural way into amalgams of selection or rejection probabilities.

VI. An Example

To illustrate the above arguments, an example is offered. Consider a project set with usefulness indices:

$$U_1 = \ln(5) + e_1$$

$$U_2 = \ln(4) + e_2$$

$$U_3 = 0 + e_3 .$$

For this project set, the accompanying Table gives ranking probabilities associated with various distributions for (e). Probabilities for the extreme value distributions are computed from equations (9) and (19) respectively.

VII. Extensions

Thusfar, analysis in this note has been confined to probabilistic selection or rejection of individual projects from a given project set, or exhaustive rankings of this project set. Other regimes of choice are of course technically possible, and it is useful to consider two ot these forms. First, denote a partial ranking of (j) out of (n) projects as (r;j,n). When the IIDRU model holds, the probability of a given partial ranking may be expressed as:

$$(27) \qquad Pr(r;j,n) = Pr(U(y_1) > \ldots > U(y_j) > \max(U(y_{j+1}) \ldots U(y_n)))$$

When the error term for (2) has the extreme value distribution, equation (27) becomes:

$$(28) \qquad Pr(r;j,n) = \frac{\prod_{i=1}^{j} \exp(d^t w_i)}{\prod_{i=1}^{j} \left(\sum_{k=i}^{n} \exp(d^t w_k) \right)}$$

which decomposes into:

$$(29) \qquad Pr(r;j,n) = Pr\,S(r_1; r_1, r_2, \ldots r_n)$$
$$* \; Pr\,S(r_2; r_2, r_3, \ldots r_n)$$
$$* \ldots * \; Pr(r_j; r_j, r_{j+1}, \ldots r_n)$$

A second useful regime for choice is the categorical ranking where projects are segregated into (c) categories of <u>predetermined</u> size. Under the IIDRU model, the probability of a categorical ranking with category sizes $(j_1, j_2, \ldots j_c)$ becomes:

$$(30) \qquad Pr(r; j_1, j_2, \ldots j_c) = Pr(U(y_1), U(y_2), \ldots U(y_{j_1})$$
$$> U(y_{j_1+1}), U(y_{j_1+2}), \ldots U(y_{j_1+j_2})$$
$$> \ldots > U(y_{n-j_c}), U(y_{n-j_c+1}), \ldots U(y_n))$$

Unfortunately, this probability does not decompose into a product of selection or rejection probabilities when random utilities obey any of the distributional laws in this note.

NOTES

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1. Equation (2) introduces a dual notation $V(z_j)$ and $(b x_j)$ for the nonrandom component of the utility function. While awkward at times, this notation appears necessary for the purposes of this essay. Most of the theoretical results cited depend on the fully general form $V(z_j)$, while the applications for empirical work require the linear approximation $(b^t x_j)$.

2. The project numbering provides one sequence of the project (z_j), while the agency ranking provides a different sequence (y_k).

3. Again a dual notation is maintained with $V(y_j)$ and $(b^t w_j)$ respectively denoting the general nonrandom component of utility, and its linear approximation.

4. To interpret (5), define $f_j(v)$ as $f(v - b^t x_j)$. Then for the case of three projects, the probability that project (j) will be superior in random utility, hence chosen becomes:

$$\Pr S(j;j,k,l) = \int_{-\infty}^{\infty} f_j(v) F_k(v) F_l(v)\,dv$$

   Thus for each value of the domain (v), the product of the probability that $(U_j = v)$ with the probability that $(U_k < v)$ and $(U_l < v)$ is considered. The integral over all such values (v) gives the appropriate choice probability.

5. The interpretation of (6) is also straightforward. Now define $f_j(v)$ to be $f(v - b^t w_j)$. Then the probability of a ranking for, say, three projects may be written:

TABLE

This table presents probabilities (as in (6)) for the six possible exhaustive rankings of three projects with random utilities, where nonrandom utility components are a given in the text and random utility components are independently, identically distributed as shown.

	error distributions	
rankings	A	B
(123)	.400	.383
(132)	.100	.144
(213)	.333	.307
(231)	.066	.110
(312)	.055	.029
(321)	.044	.027

Error Distributions:

A: extreme value; equation (6)

B. reverse extreme value; equation (14)

$$\Pr(r_1, r_2, r_3) = \int_{-\infty}^{\infty} f_1(v_1) \int_{-\infty}^{v_1} f_2(v_2) \int_{-\infty}^{v_2} f_3(v_3)\, dv_3\, dv_2\, dv_1$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{v_1} \int_{-\infty}^{v_2} f_1(v_1)\, f_2(v_2)\, f_3(v_3)\, dv_3\, dv_2\, dv_1$$

Note that this probability is simply the integral of the independent densities bounded by $v_1 > v_2 > v_3$. The general case is simply an extension of this example.

6. For a discussion of rejection probabilities outside the context of IIDRU models, see the reference by Marly.

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Faculty Working Papers

RE-EXAMINING THE WELFARE LOSS DUE TO MONOPOLY

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#670

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Notes

[1]We required the firms to be listed during the entire sample period. The Center for Security Price Research (CRSP) monthly tape was used to select NYSE listed firms. A firm was considered listed if it had monthly stock returns available for the entire sample period.

[2]The absolute percentage error is computed as the average of $\left|\frac{\text{Actual EPS - Predicted EPS}}{\text{Actual EPS}}\right|$. Since this error metric can be explosive when the denominator approaches zero we truncated errors in excess of ten to a value of ten. This operation was done for a very small percentage of the cases.